Maple lab 1 is due next Monday (6/8/15)

Mathematical model with ODEs

Caution: all math models have limitation.

Example 1: Newton's law of cooling

Temperature of an object changes at a rate proportional to the difference between its temperture and that of its surroundings

Only applies to convective heat transfer. Not to be used for other kind of heat transfered.

Rate of change $= \frac{dT}{dt}$, Set T as the temperature of the object. Set T_a temp. of the surrounding.

$$\frac{dT}{dt} = -k(T - T_a)$$

Suppose the temp. of a cup of coffee obeys NLC. If the coffee has a temprature of 200°F when freshly poured. one minute later it has cooled to 190°F in a room of temperature 70 °F. Determine when the coffee reach 150 °F.

modeling.

$$\left.\begin{aligned} \frac{dT}{dt} &= -k(T-T_a) \\ T(0) &= 200\,°F \\ T(1) &= 190\,°F \\ T_a &= 70\,°F \end{aligned}\right\} \quad \begin{aligned} T(\tau) &= 150\,°F \\ \tau &= ? \end{aligned}$$

Solve

Separating the variable and solve the ODE

$$T(t) = T_a + Ce^{-kt}$$

Put in the values of $t=0$ & $t=1$

$$T(0) = 70 + Ce^{0} = 200$$

$$T(1) = 70 + Ce^{-k} = 190.$$

$$\Rightarrow \quad C = 130.$$

$$130e^{-k} = 120 \Rightarrow e^{-k} = \frac{12}{13} \Rightarrow -k = \ln\frac{12}{13}$$

Recall: $e^a = b, \Leftrightarrow a = \ln b$

$$c^a = b \Leftrightarrow a = \log_c b$$

$$\Rightarrow k = -\ln\frac{12}{13} \qquad a < 1 \Rightarrow \ln a < 0$$

$$T(t) = 70 + 130 e^{\ln\frac{12}{13} t}$$

Recall: change of base: $e^{(\ln a)t} = a^t$

$$= 70 + 130\left(\frac{12}{13}\right)^t$$

$$T(\tau) = 150 \Rightarrow 150 = 70 + 130\left(\frac{12}{13}\right)^\tau$$

$$80 = 130\left(\frac{12}{13}\right)^\tau$$

$$\frac{8}{13} = \left(\frac{12}{13}\right)^\tau$$

$$\ln\frac{8}{13} = \tau \cdot \ln\frac{12}{13}$$

$$\tau = \frac{\ln 8/13}{\ln 12/13} \cancel{=} \frac{\ln 8}{\ln 12}$$ — abuse of alg.

$$= \frac{\ln 8 - \ln 13}{\ln 12 - \ln 13} \doteq 6.066 \quad (\text{min})$$

Example 2: Pollution Model

A newly built factory emits mercury pollutants continuously with a rate 50 grams per day to a fresh water lake.

Suppose the volume of water in the lake is constantly 1,000,000 m^3. Every day $\underbrace{20{,}000\ m^3}_{\text{constant}}$ of the water is refreshed. Find the time it takes for the water to be undrinkable (concentration of Hg arrives at $0.002\,mg/L$)

Modeling: P pollutant Take Δt small period of time

Change of pollutant ΔP within Δt.

γ rate of emission. $\gamma = 0.05\ kg/day$

V volume of water in the lake $V = 10^6\ m^3$

W refreshing rate: $W = 2\times 10^4\ m^3/day$

Within Δt, $\gamma\Delta t$ pollutants into the lake

with the assumption water is well mixed every moment

$\underbrace{\left(\frac{P}{V}\right)}_{\text{concentration}} \overbrace{W\Delta t}^{\text{water leaving the lake}}$ pollutants out of the lake

$$\Delta P = \gamma\Delta t - \frac{P}{V}W\Delta t \Rightarrow \frac{\Delta P}{\Delta t} = \gamma - \frac{P}{V}W$$

$\Delta t \to 0$ $$\frac{dP}{dt} = \gamma - \frac{P}{V}W$$

Separating variables and solve the ODE

$$\frac{dP}{\gamma - (W/V)P} = dt \Rightarrow -\frac{V}{W}\cdot \ln\left|\gamma - \frac{W}{V}P\right| = t + C$$

$$\Rightarrow \gamma - \frac{W}{V}P = Ce^{-\frac{W}{V}t} \Rightarrow P = \frac{V}{W}\left(\gamma - Ce^{-\frac{W}{V}t}\right)$$

$$P(t) = \frac{V}{W}\left(\gamma - Ce^{-\frac{W}{V}t}\right)$$

$t=0$ time when the factory is built. $P(0)=0$

Find $P(\tau)$ s.t. $\frac{P(\tau)}{V} = 0.002\ mg/L = 2\times 10^{-6}\ kg/m^3$

$0.002\ mg/L = 0.002 \cdot 10^{-6}\ kg/10^{-3}\ m^3 = 0.002\times 10^{-3}\ kg/m^3$
$= 2\times 10^{-6}\ kg/m^3$

$$P(t) = \frac{10^6}{2\times 10^4}\left(0.05 - Ce^{-\frac{2\times 10^4}{10^6}t}\right)$$

$$= 50\left(0.05 - Ce^{-0.02t}\right)$$

$P(0)=0 \Rightarrow C = 0.05$

$$\Rightarrow P(t) = 2.5\left(1 - e^{-0.02t}\right)$$

$\frac{P(\tau)}{V} = 2\times 10^{-6} \Rightarrow P(\tau) = 2.$

$$\Rightarrow 2 = 2.5\left(1 - e^{-0.02t}\right) \Rightarrow \frac{4}{5} = 1 - e^{-0.02t}$$

$$\Rightarrow e^{-0.02t} = \frac{1}{5} \Rightarrow -0.02t = \ln\frac{1}{5}$$

$$\Rightarrow t = -\frac{1}{0.02}\ln\frac{1}{5} \doteq 80.5$$

Problem: Is it possible to control the emission of pollutant s.t. water stay drinkable forever.

Example 3: Compound Interest.

I = Annual investment $\qquad I = 12{,}000$ /yr.

r = Annual Interest Rate $\qquad r = 1\% = 0.01$

$C(t)$ — Capital at time t

$C(0) = 0.$

$C(8) = ?$

within a small time Δt, capital changes ΔC

$$\Delta C = I\,\Delta t + r\,C(t)\,\Delta t$$

$$\frac{\Delta C}{\Delta t} = I + rC$$

$\Delta t \to 0.$ $\qquad \dfrac{dC}{dt} = I + rC$

Separating the variables and solve the ODE

$$\ln|I + rC| = rt + K$$

$$rC(t) = Ke^{rt} - I$$

K → arbitrary constant.

$C(0)=0 \Rightarrow K-I=0 \Rightarrow K=I$

$C(t)=\frac{I}{r}(e^{rt}-1)$

$I=12{,}000, \quad r=1\%=0.01.$

$C(8)=\frac{12{,}000}{0.01}(e^{0.08}-1)=99944.48.$

If without saving account, $12000\times 8=96000$

Interest $=3944.48.$ MEAN!

Example: Peskin's Loan APR $=r=5\%$

Borrow $B=20{,}000$ /yr

Debt $D(t)$. $D(0)=0$

$D(4)=?$

$$\frac{dD}{dt}=rD+B$$

Separating the variable: $\frac{dD}{rD+B}=dt$

$\frac{1}{r}\ln|rD+B|=t+C$

$D=(Ce^{rt}-B)\frac{1}{r}$

$D(0)=0 \Rightarrow C=B \Rightarrow D(t)=\frac{B}{r}(e^{rt}-1)$

$B=20{,}000, \quad r=0.05, \quad D(4)=\frac{20000}{0.05}(e^{0.2}-1)=88{,}561.10$

Interest over 4 years = 8,561.10.

Payment $P = 6{,}000$ /yr continuously.

$$\frac{dD}{dt} = rD - P, \quad D(0) = 88561.10$$

$$D(\tau) = 0, \quad \tau = ?$$

Solve the ODE $\quad D(t) = \frac{1}{r}\left(Ce^{rt} + P\right)$

$$D(0) = 88561.10 \Rightarrow \frac{1}{0.05}(C + 6000) = 88561.10.$$

$$\Rightarrow C = -1571.94$$

$$D(\tau) = \frac{1}{0.05}\left(-1571.94e^{0.05\tau} + 6000\right) = 0$$

$$1571.94\, e^{0.05\tau} = 6000$$

$$\tau = \frac{1}{0.05} \ln \frac{6000}{1571.94} = 26.79 \text{ yr.}$$

Attendance Quiz: If after 3 years. Mr. Qi got a promotion. allowing him to save 30,000 /yr to pay the debt, how many more time does it take for him to pay off the debt?

$D(3) = 83474.02.$

Falling Object:

Newton's second law: $m\frac{dv}{dt} = F$ the total force

For a falling object, it's subject to two forces:

Gravity $= mg$

Air resistant $= -kv^2$.

ODE: $m\frac{dv}{dt} = mg - kv^2$

Separate the variables

$$\frac{dv}{mg - kv^2} = \frac{1}{m}dt$$

Recall: $\frac{1}{a - v^2} = \left(\frac{1}{\sqrt{a}+v} + \frac{1}{\sqrt{a}-v}\right)\frac{1}{2\sqrt{a}}$

$$\frac{1}{mg - kv^2} = \frac{1}{k}\,\frac{1}{\frac{mg}{k} - v^2} = \frac{1}{k}\left(\frac{1}{\sqrt{mg/k}+v} + \frac{1}{\sqrt{mg/k}-v}\right)\frac{1}{2\sqrt{mg/k}}$$

$$= \frac{1}{2\sqrt{mgk}}\left(\frac{1}{\sqrt{mg/k}+v} + \frac{1}{\sqrt{mg/k}-v}\right)$$

$$\int \frac{dv}{mg - kv^2} = \frac{1}{2\sqrt{mgk}}\ln\left|\frac{\sqrt{mg/k}+v}{\sqrt{mg/k}-v}\right| = \frac{1}{m}t + C$$

$$\frac{\sqrt{mg/k}+v}{\sqrt{mg/k}-V} = Ce^{2\sqrt{\frac{gk}{m}}\,t}$$

$$v + \sqrt{mg/k} = Ce^{2\sqrt{\frac{gk}{m}}t}\left(\sqrt{mg/k} - v\right)$$

$$v(t) = \frac{\sqrt{mg/k}\left(Ce^{2\sqrt{\frac{gk}{m}}t} - 1\right)}{Ce^{2\sqrt{\frac{gk}{m}}t} + 1}$$

Example: $mg = 192\ \text{lb}, \quad k = 12, \quad g = 32\ \text{ft}/\text{s}^2$

$$\sqrt{mg/k} = \sqrt{192/12} = 4$$

$$\sqrt{\frac{gk}{m}} = \sqrt{\frac{32 \times 12}{192/32}} = \frac{32}{4} = 8$$

(a) $$v(t) = \frac{4\left(Ce^{16t} - 1\right)}{Ce^{16t} + 1}$$

$$v(0) = 0 \Rightarrow \frac{4(C-1)}{C+1} = 0 \Rightarrow C = 1$$

$$v(t) = \frac{4\left(e^{16t} - 1\right)}{e^{16t} + 1} = 4 - \frac{8}{e^{16t} + 1}$$

$$d(t) = 4t - \int_0^t \frac{8}{e^{16t} + 1}\,dt$$

$u = e^{16t} + 1. \qquad du = 16e^{16t}\,dt = 16(u-1)\,dt$

$$\int \frac{8}{e^{16t}+1}\,dt = \int \frac{du}{2(u-1)u} = \frac{1}{2}\int\left(\frac{1}{u-1} - \frac{1}{u}\right)du$$

$$= \frac{1}{2}\ln\frac{u-1}{u} + C = \frac{1}{2}\ln\frac{e^{16t}}{e^{16t}+1} + C$$

$$= 8t - \frac{1}{2}\ln(e^{16t}+1) + C$$

$$d(t) = -4t + \frac{1}{2}\ln\left(e^{16t}+1\right)$$

(b) $\lim\limits_{t\to\infty} v(t) = 4 \quad ft/s$

(c) $d(10) = -40 + \frac{1}{2}\ln\left(e^{160}+1\right) \doteq 40$

$$v(10) = 4 - \frac{8}{e^{160}+1} \doteq 4$$

4.2. Air resistant $= kv^2$

$$\frac{1}{2}A = k\cdot 20^2 \Rightarrow A = 800k$$

Landing velocity $= \sqrt{\frac{mg}{k}} = 16 \Rightarrow k = \frac{mg}{16^2} = \frac{192}{16^2}$

$$= 4.5.$$

$$\Rightarrow A = 800 \times 4.5 = 3600 \quad ft^2.$$

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